Subgraph Federated Learning via Spectral Methods

Alexandre Graell i Amat

Chalmers University of Technology, Gothenburg, Sweden

in collaborarion with Javad Aliakbari and Johan Östman

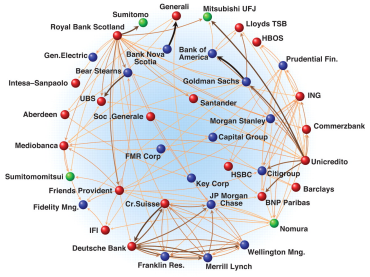
Workshop on Information Theory for Future Networks

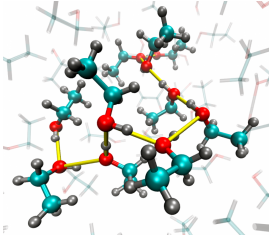
Karlsruhe Institute of Technology, Karlsruhe, Germany

March 3, 2026

CHALMERS

Graph-structured data

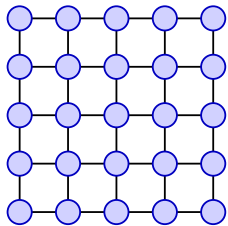




Graph neural networks

- Traditional ML designed for sequences (text, speech) or grid data (images)

fixed order, local context



regular structure, spatial locality

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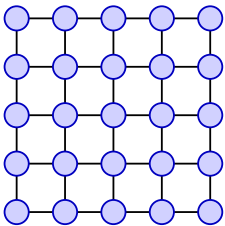

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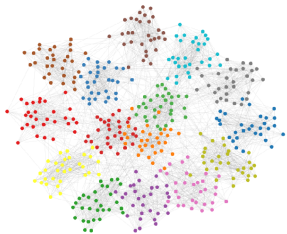

irregular
no canonical ordering

GNNs: Learn node representations that encode both features and structural information.

Applications

Node embeddings can be used for various machine learning tasks:

- Node-level tasks: Node classification
- Link-level tasks: Link prediction
- Graph-level tasks: Graph classification

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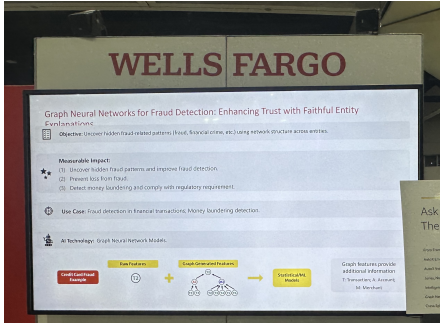
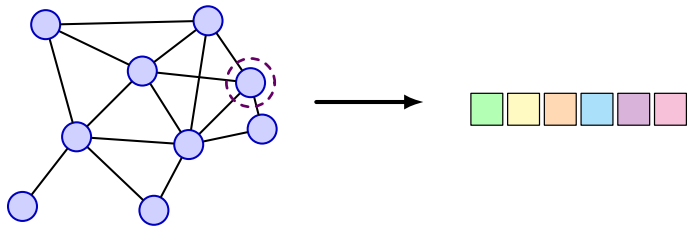


NeurIPS 2025

Node embeddings

Goal: Given $\mathcal{G}$, learn a low-dimensional embedding $\boldsymbol{z}_v \in \mathbb{R}^d$ for each node $v \in \mathcal{V}$



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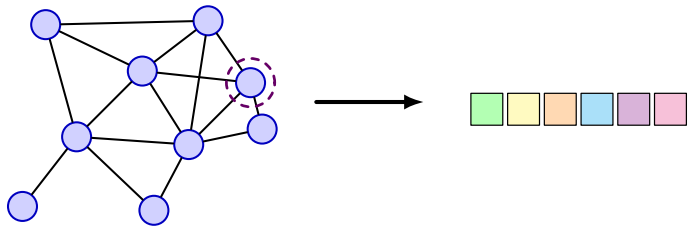

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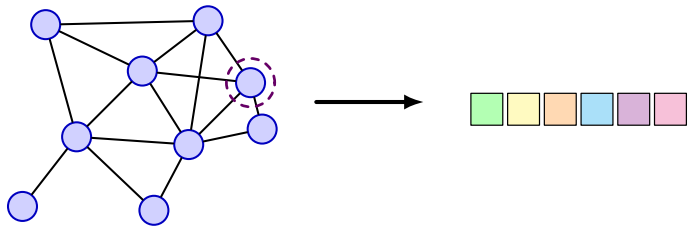

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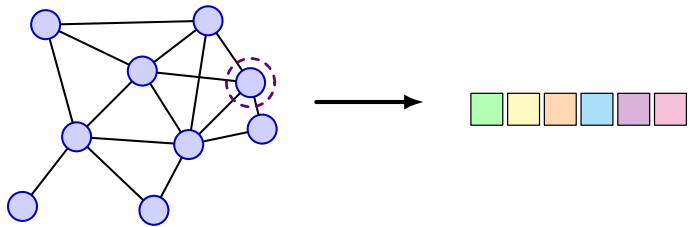

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- Node similarity in the graph $\longrightarrow$ similarity in embedding space

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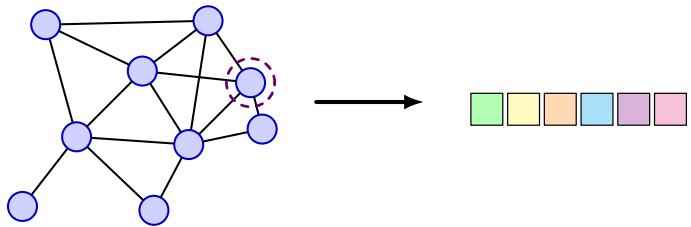

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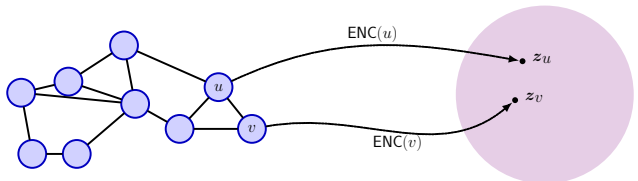
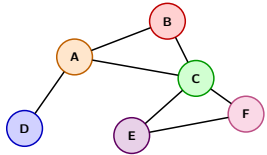


Graph neural networks



Core principle: Compute node representation by aggregating information from neighbors via message passing.

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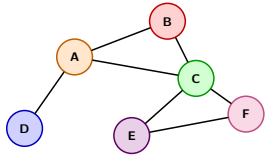

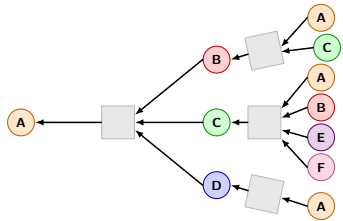

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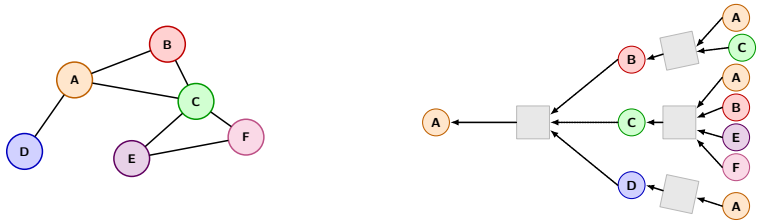

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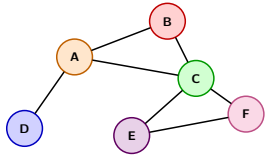
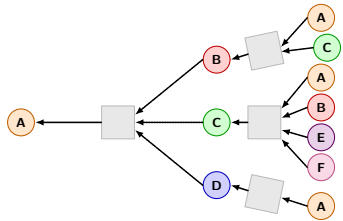
Two key operations at each layer:

Graph neural networks





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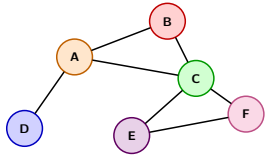

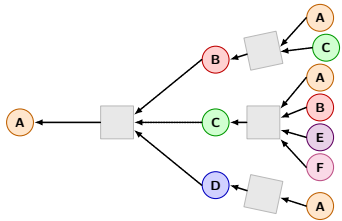

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Two key operations at each layer:

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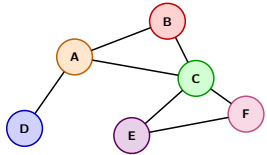

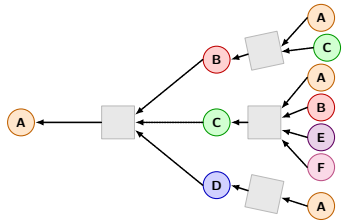

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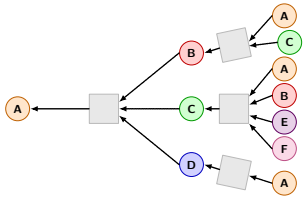
- GNN depth determines receptive field: L layers $\rightarrow$ L-hop neighborhood

Neural message passing

Notation:

- $\boldsymbol{x}_v$: feature vector of node v
- $\boldsymbol{h}_v^{(\ell)}$: embedding of node v at layer ℓ
- $\boldsymbol{h}_v^{(0)} = \boldsymbol{x}_v$



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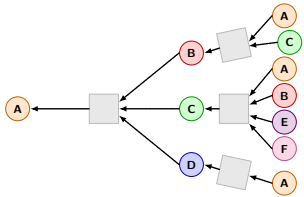

At each layer ℓ, two steps:

1. Aggregation step: Embeddings from neighbors passed to v and combined

$$\boldsymbol{m}_v^{(\ell)} = \text{Agg}\left(\left\{\boldsymbol{h}_u^{(\ell)}, u \in \mathcal{N}(v)\right\}\right)$$

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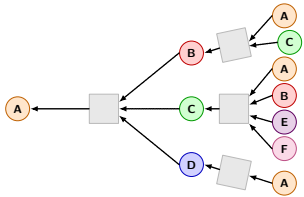

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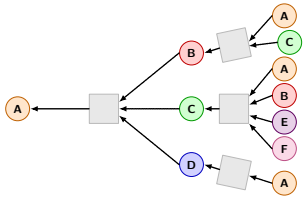

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Aggregate and update are arbitrary differentiable functions (neural networks).

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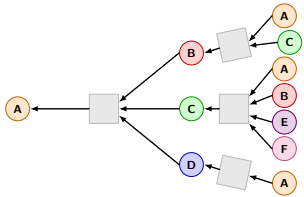

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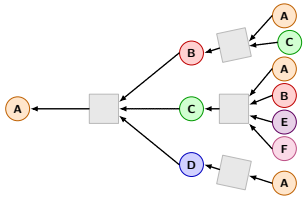

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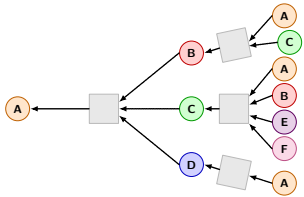

- Node embedding at layer $\ell + 1$:

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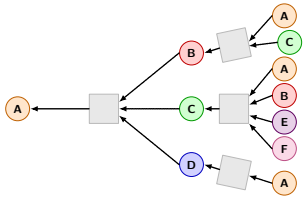

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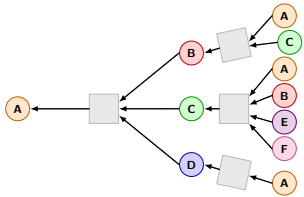

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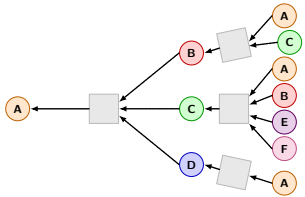

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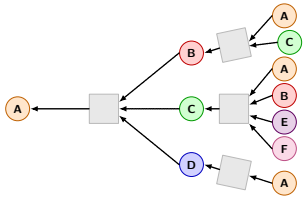

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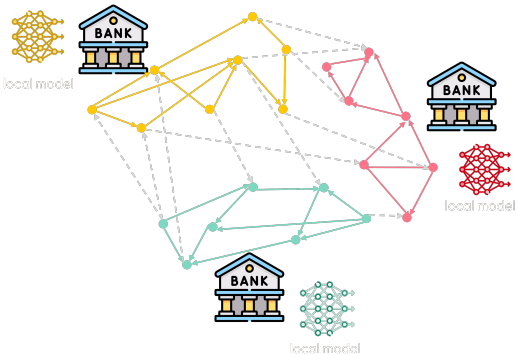
Important: Same parameters $\boldsymbol{W}^{(\ell)}, \boldsymbol{B}^{(\ell)}$ shared across all nodes!

The need of federated learning

How do we train a machine learning model on data distributed across several clients while preserving data privacy?



Federated learning

central server

Federated learning [1]:

[1] H. B. McMahan, E. Moore, D. Ramage, S. Hampson, and B. Agüera y Arcas, "Communication- efficient learning of deep networks from decentralized data," AISTATS, 2017.

Federated learning

Federated learning [1]:

- Training a global model in a distributed fashion across clients

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Federated learning [1]:

- Training a global model in a distributed fashion across clients
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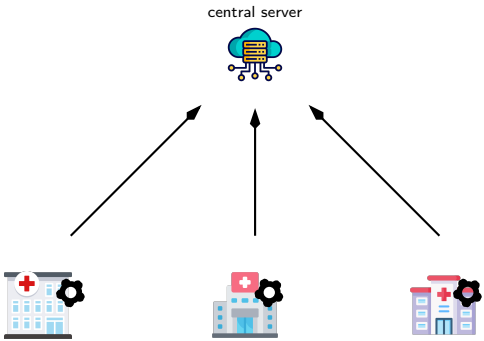

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- Clients send model updates to central server

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- Training a global model in a distributed fashion across clients
- Clients train models locally
- Clients send model updates to central server
- Central server aggregates received models and broadcasts result back to clients

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- Clients send model updates to central server
- Central server aggregates received models and broadcasts result back to clients

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Federated learning

Federated learning [1]:

- Training a global model in a distributed fashion across clients
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- Multiple rounds

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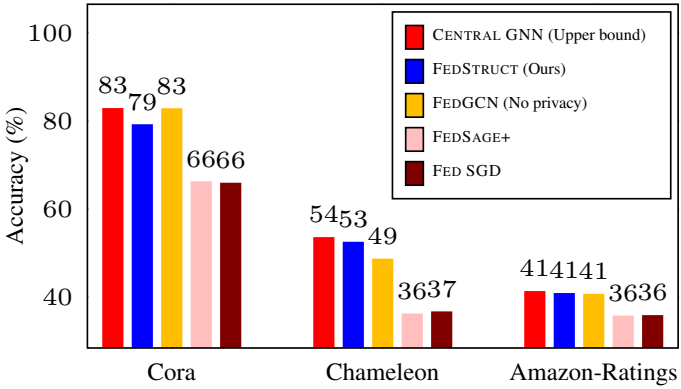
Results



- 10 clients

Subgraph federated learning

Interconnected subgraphs: Interconnections play critical role!

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But no privacy!

FedStruct

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- $\boldsymbol{h}_v$: Node feature embeddings, computed locally, $\boldsymbol{h}_v = f_{\boldsymbol{\theta}_\mathsf{f}}(\boldsymbol{X}_i, \mathcal{E}_i, v)$
- $\boldsymbol{z}_v$: Node structure embeddings (encode nodes' structural information; clients need to collaborate)

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Node structure embeddings

- To generate NSEs, FedStruct relies on node structure features (encapsulate structural information about the nodes)

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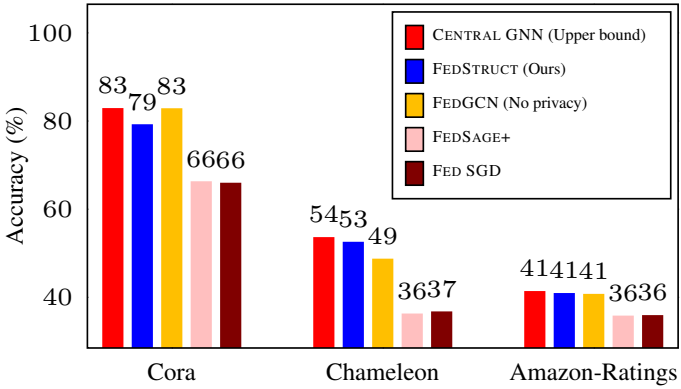

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communication cost, no formal privacy guarantees

FedLap: Subgraph federated learning via spectral methods

J. Aliakbari, J. Östman, A. Panahi, and A. Graell i Amat, "Subgraph Federated Learning via Spectral Methods," NeurIPS, 2025.

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Core idea: Leverage structural information through Laplacian smoothing.

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Recall (FedStruct):

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FedLap: Constrains $\boldsymbol{S}$ introducing a Laplacian smoothing regularizer:

$$\mathcal{L}(\boldsymbol{\theta}) = \mathcal{L}_\text{c}(\boldsymbol{\theta}) + \lambda_\text{reg} \frac{\text{Tr}(\boldsymbol{S}^\top \boldsymbol{L}_\mathcal{G} \boldsymbol{S})}{\text{Tr}(\boldsymbol{S}^\top \boldsymbol{S})}, \qquad \text{where} \quad \boldsymbol{L}_\mathcal{G} = \boldsymbol{D} - \boldsymbol{A}$$

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- The regularizer encourages neighboring nodes to have similar structure embeddings
- Avoids overfitting on training nodes

Lower communication, better privacy.

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Intuition: Large λ_j (high-frequency components) are penalized more $\Rightarrow$ energy moves to small λ (smooth) components

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FedLap+: Computationally efficient ($\mathcal{O}(nr^2)$), scalable (for large, sparse graphs), privacy-preserving.

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Approach: A membership inference formulation.

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- Applies the likelihood ratio test:

$$\text{LLR}_{u,v} = \log \frac{P(\boldsymbol{U}|\boldsymbol{A}_{uv} = 1)}{P(\boldsymbol{U}|\boldsymbol{A}_{uv} = 0)}$$

(optimal by Neyman–Pearson lemma)

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Theorem:

$$H_1 : \quad \text{LLR}_{u,v} \sim \mathcal{N}\left(\frac{1}{2}\alpha, \alpha\right), \quad H_0 : \quad \text{LLR}_{u,v} \sim \mathcal{N}\left(-\frac{1}{2}\alpha, \alpha\right)$$

$$D_{\text{KL}}\left(\Pr(\text{LLR}_{u,v} \mid H_1) \,\middle\|\, \Pr(\text{LLR}_{u,v} \mid H_0)\right) \approx \alpha/2, \qquad \alpha = \frac{r}{np(1-p)}$$

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- For practical values of r, n, and p,

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where precision$= \text{TP}/(\text{TP} + \text{FP})$, recall$= \text{TP}/(\text{TP} + \text{FN})$, γ is the decision threshold

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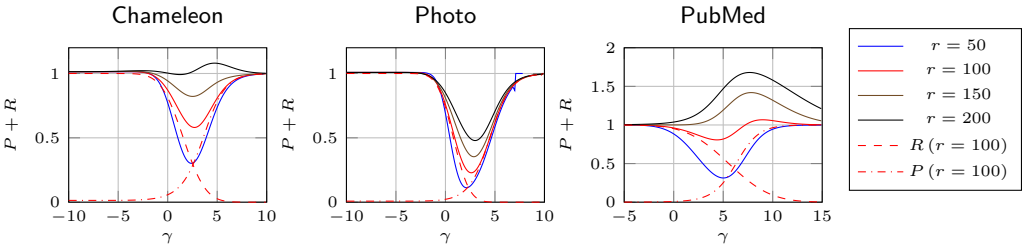

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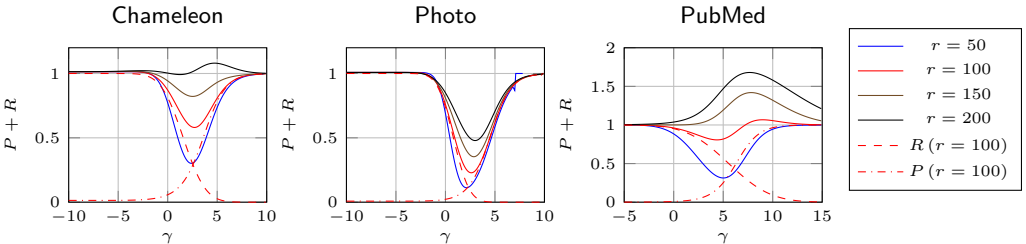

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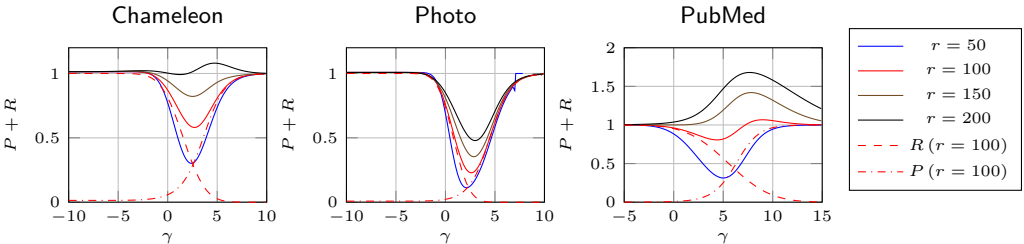

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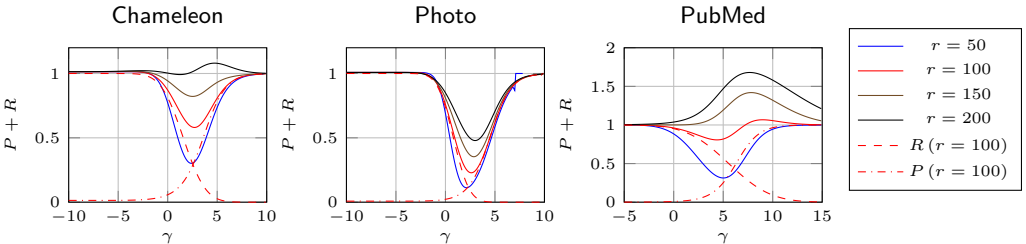

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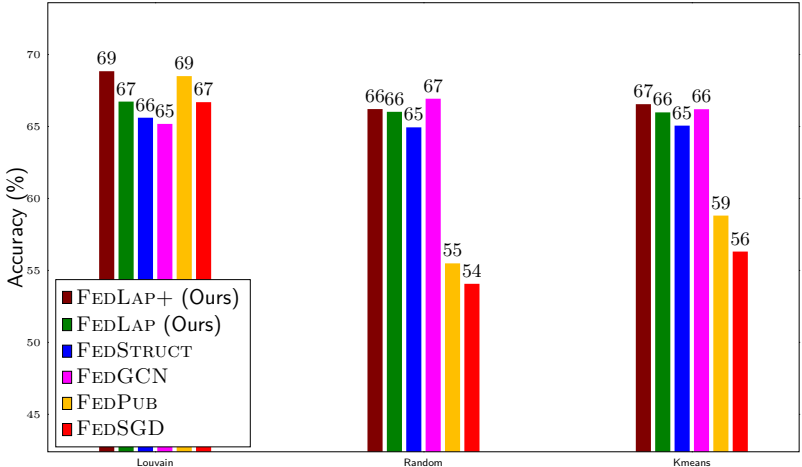
An unrealistically strong adversary cannot reliably recover structural information.

Experimental results



- Dataset: ovbn-arxiv (169.343 nodes, 1.166.243 edges). 10% training ratio

Conclusion

FedLap: The state-of-the-art method for subgraph federated learning.

- Performance close to centralized setting
- Reduced the communication compared to FedStruct
- Strong privacy guarantees (membership inference-based analysis)
- Scalable and robust across different graph topologies, including heterophilic graphs

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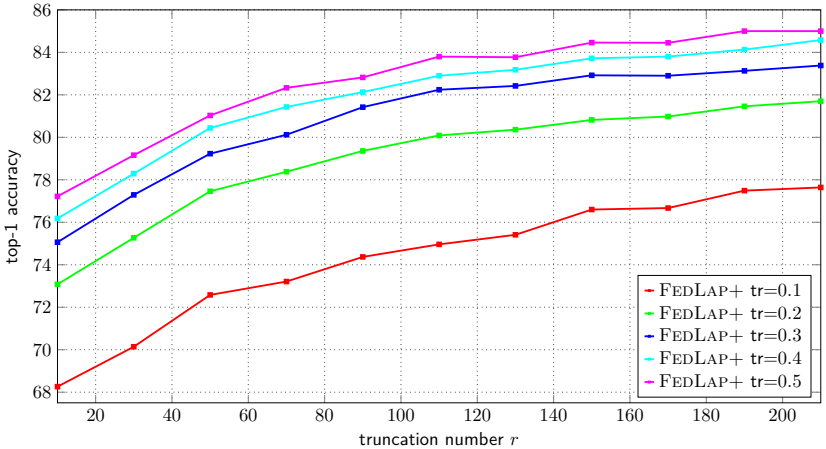
What's next?

- Extension to dynamic graphs
- Gang detection

Experimental results: Truncation parameter r



- Dataset: Cora (2708 nodes, 10556 edges)

FedLap framework