An Orthogonal Approximate Message Passing Framework for Multi-User Communications

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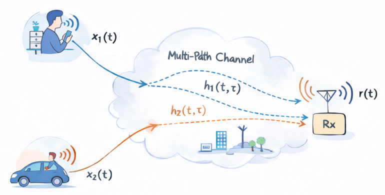
Multi-user communications



$$h_u(t,\tau) = \sum_{p \leq P_u} \alpha_{u,p}\, \mathrm{e}^{\mathrm{j}2\pi\nu_{u,p}t}\, \delta(\tau - \tau_{u,p})$$

- Let $\mathbf{s}_u \in \mathbb{C}^{N_u}$ denote the input signal vector of user u, possibly encoded, e.g., using SPARC–LDPC) [Ebert, Chamberland, Narayanan, 23]
- We consider random precoding [Müller, Guo, Moustakas, 08], [Liu, Chi, Huang, 25] where the transmitted signal is

$$\mathbf{x}_u = \mathbf{\Xi}_u \mathbf{s}_u$$

- $\mathbf{\Xi}_u$ is right-unitarily invariant; for example, $\mathbf{\Xi}_u$ can be random semi-unitary (i.e., $\mathbf{\Xi}_u = \mathbf{Haar}_{N_u}(1{:}L, :)$).

Multi-user communications

After cyclic prefix removal, the received signal can be written as

$$\mathbf{y} = \sum_{u} \underbrace{\left(\mathbf{U}_{\psi} \mathbf{H}_{0,u} \mathbf{U}_{\psi}^{\dagger}\right)}_{\doteq \mathbf{H}_u} \mathbf{\Xi}_u \mathbf{s}_u + \mathbf{n}$$

- $\mathbf{U}_{\psi} \in \mathbb{C}^{L \times L}$ is a unitary matrix determined by the waveform, e.g.,

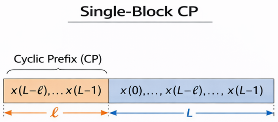
$$\mathbf{U}_{\psi} = \begin{cases} \mathbf{I}_n \otimes \mathrm{FFT}_M & \text{OFDM} \\ \mathrm{FFT}_n \otimes \mathbf{I}_M & \text{OTFS} \end{cases}$$

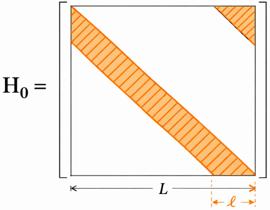
- $\mathbf{H}_{0,u} \in \mathbb{C}^{L \times L}$ denotes the effective discrete-time channel response matrix, constructed from the physical path parameters $(\alpha_{u,p}, \nu_{u,p}, \tau_{u,p})$ where

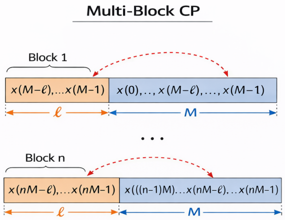
$$h_u(t, \tau) = \sum_{p \leq P_u} \alpha_{u,p} \, e^{j2\pi \nu_{u,p} t} \, \delta(\tau - \tau_{u,p})$$

Role of cyclic prefix

Cyclic prefix (CP) is appended to the transmitted signal $\mathbf{x} \equiv \mathbf{x}_u \in \mathbb{C}^{L=nM}$.







$$\mathbf{H}_0 = \begin{bmatrix} \mathbf{H}_0^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_0^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_0^n \end{bmatrix}$$

Generic model

Consider the observation model

$$\mathbf{y} = \sum_u \mathbf{A}_u \mathbf{s}_u + \mathbf{n}$$

- Each $\mathbf{A}_u \in \mathbb{C}^{L \times N_u}$ is mutually distinct (across u) and right-unitarily invariant, i.e.,

$$\mathbf{A}_u = \mathbf{U}_u \mathbf{D}_u \mathbf{O}_u$$

  where $\mathbf{O}_u$ is an arbitrary Haar unitary.
- Each $\mathbf{s}_u$ is an independent random vector with statistically coupled entries.

What's new?

Define the joint (across users) variables

$$\mathbf{s} \doteq [\mathbf{s}_1^\top, \mathbf{s}_2^\top, \ldots, \mathbf{s}_U^\top]^\top$$
$$\mathbf{A} \doteq [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_U]$$

Hence, we have the classical observation model

$$\mathbf{y} = \mathbf{A}\mathbf{s} + \mathbf{n}$$

$\mathbf{A}$ is not right-unitarily invariant!

Recall that in our context $\mathbf{A}_u \equiv \mathbf{H}_u \mathbf{\Xi}_u$.

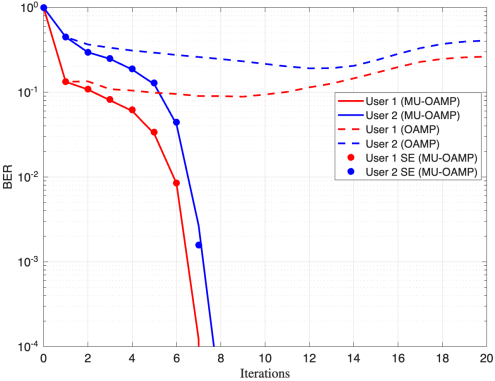
- AMP (approximate message passing) typically fails.
- OAMP [Ma, Ping, 17]–also known as VAMP [Rangan, Scheiter, Fletcher,19]–leads to poor results as statistical asymmetry among the users (i.e. $\mathbf{A}_u \mathbf{s}_u \not\sim \mathbf{A}_{u'} \mathbf{s}_{u'}$) increases.
- Complete statistical symmetry manifests an asymptotic universality.

MU-OAMP vs. OAMP

Let us consider two-user case such that

$$\mathbf{A}_1\mathbf{s}_1 \sim \sqrt{\rho}\,\mathbf{A}_2\mathbf{s}_2$$

where we introduce a simple asymmetry over the users with the scaling ρ.



Time-varying channels, i.i.d. QPSK signals, $N_u = L = n \times M = 2^4 \times 2^8, \mathrm{SNR} = 13$ dB, $\rho = 6$ dB.

Part-I: Algorithmic Aspects

Expectation consistency (EC) method [Opper, Winther,05]

- Consider $\mathbf{y} = \mathbf{As} + \mathbf{n}$ such that the posterior factors as

$$p(\mathbf{s}|\mathbf{y}, \mathbf{A}) \propto p_0(\mathbf{s}) \underbrace{\mathcal{CN}(\mathbf{y}|\mathbf{As}, \sigma^2 \mathbf{I}_L)}_{\doteq p_1(\mathbf{s})}$$

- The rule of EC: $p(s_i|\mathbf{y}, \mathbf{A})$ is approximated to Gaussian pdfs $q(s_i)$ s.t.

$$q(\mathbf{s}) \propto \underbrace{\mathcal{CN}(\mathbf{s}|\mathbf{f}, \mathrm{diag}(\boldsymbol{\lambda}))}_{\doteq q_0(\mathbf{s})} \underbrace{\mathcal{CN}(\mathbf{s}|\mathbf{r}, \mathrm{diag}(\boldsymbol{\tau}))}_{\doteq q_1(\mathbf{s})}$$

- $(\mathbf{f}, \boldsymbol{\lambda})$ and $(\mathbf{r}, \boldsymbol{\tau})$ are obtained by iteratively solving the EC conditions

$$\mathbb{E}[\{g(\mathbf{s})\}]_{q_0 \cdot p_1} = \mathbb{E}[\{g(\mathbf{s})\}]_{q_0 \cdot q_1} = \mathbb{E}[\{g(\mathbf{s})\}]_{p_0 \cdot q_1}$$

  for $p(\mathbf{s}) \propto f_0(\mathbf{s}) f_1(\mathbf{s})$ we write $\mathbb{E}[(\cdot)]_{f_0 \cdot f_1} \triangleq \int (\cdot) p(\mathbf{s}) \mathrm{d}\mathbf{s}$
- Without any restrictions on q, we have $\{g(\mathbf{s})\} = \{s_i, |s_i|^2\}$.

Proposed approach

$$\mathbb{E}[\{g(\mathbf{s})\}]_{q_0 \cdot p_1} = \mathbb{E}[\{g(\mathbf{s})\}]_{q_0 \cdot q_1} = \mathbb{E}[\{g(\mathbf{s})\}]_{p_0 \cdot q_1}$$

- Consider the following restriction for variances $\boldsymbol{\theta} \in \{\boldsymbol{\lambda}, \boldsymbol{\tau}\}$ as

$$\boldsymbol{\theta} = \begin{cases} (\theta_1, \theta_2, \cdots, \theta_N) & \text{EP [Minka, 01]} \\ (\theta_1 \mathbf{1}_{N_1}, \theta_2 \mathbf{1}_{N_2}, \cdots, \theta_U \mathbf{1}_{N_U}) & \text{Proposed Approach} \\ \theta_u \mathbf{1}_N & \text{OAMP} \end{cases}$$

- Unified U-source notation: Consider the decomposition $\mathbf{As} = \sum_{u=1}^{U} \mathbf{A}_u \mathbf{s}_u$, where U can be as small or as large as desired. Then, we set

$$\{g(\mathbf{s})\} = \{\mathbf{s}, \|\mathbf{s}_u\|^2\}_{u \leq U}$$

  For example, $U = N$ gives the EP setting as $\{g(\mathbf{s})\} = \{s_i, |s_i|^2\}_{i \leq N}$ and $U = 1$ gives the OAMP setting $\{g(\mathbf{s})\} = \{\mathbf{s}, \|\mathbf{s}\|^2\}$.

Unified U-source algorithm

Start with initializations $\mathbf{f}_u^{(1)}$ and $\lambda_u^{(1)}$ and proceed for $t = 1, 2, \ldots$

$$\begin{aligned}
\mathbf{\Sigma}^{(t)} &= (\sigma^2 \mathbf{I}_L + \sum_u \lambda_u^{(t)} \mathbf{A}_u \mathbf{A}_u^\dagger)^{-1} \\
\mathbf{r}_u^{(t)} &= \frac{1}{\chi_u^{(t)}} \mathbf{A}_u^\dagger \mathbf{\Sigma}^{(t)} (\mathbf{y} - \sum_{u'} \mathbf{A}_{u'} \mathbf{f}_{u'}^{(t)}) + \mathbf{f}_u^{(t)} \\
\boldsymbol{\eta}_u^{(t+1)} &= \mathbb{E}[\mathbf{s}_u | \mathbf{r}_u^{(t)} = \mathbf{s}_u + \sqrt{\tau_u^{(t)}} \mathbf{z}] \\
\mathbf{f}_u^{(t+1)} &= \frac{\tau_u^{(t)} \boldsymbol{\eta}_u^{(t+1)} - \nu_u^{(t+1)} \mathbf{r}_u^{(t)}}{\tau_u^{(t)} - \nu_u^{(t+1)}}
\end{aligned}$$

where $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_u})$. We update the scalars as

$$\begin{aligned}
\chi_u^{(t)} &= \frac{1}{N_u} \mathrm{tr}(\mathbf{A}_u^\dagger \mathbf{\Sigma}^{(t)} \mathbf{A}_u) \\
\tau_u^{(t)} &= 1/\chi_u^{(t)} - \lambda_u^{(t)} \\
\nu_u^{(t+1)} &= \frac{1}{N_u} \sum_{i \leq N_u} \mathbb{V}[s_{u,i} | \mathbf{r}_u^{(t)} = \mathbf{s}_u + \sqrt{\tau_u^{(t)}} \mathbf{z}] \\
\lambda_u^{(t+1)} &= \tau_u^{(t)} \nu_u^{(t+1)} / (\tau_u^{(t)} - \nu_u^{(t+1)})
\end{aligned}$$

Part II: High-Dimensional Analysis of the Algorithm

State-evolution

We replace the posterior mean denoiser with a generic denoiser

$$\begin{aligned}
\boldsymbol{\Sigma}^{(t)} &= (\sigma^2 \mathbf{I}_L + \sum_u \lambda_u^{(t)} \mathbf{A}_u \mathbf{A}_u^\dagger)^{-1} \\
\mathbf{r}_u^{(t)} &= \frac{1}{\chi_u^{(t)}} \mathbf{A}_u^\dagger \boldsymbol{\Sigma}^{(t)} (\mathbf{y} - \sum_{u'} \mathbf{A}_{u'} \mathbf{f}_{u'}^{(t)}) + \mathbf{f}_u^{(t)} \\
\boldsymbol{\eta}_u^{(t+1)} &= \eta_{u,t}(\mathbf{r}_u^{(t)}) \quad \text{with} \quad \eta_{u,t} : \mathbb{C}^{N_u} \to \mathbb{C}^{N_u} \\
\mathbf{f}_u^{(t+1)} &= \frac{\tau_u^{(t)} \boldsymbol{\eta}_u^{(t+1)} - \nu_u^{(t+1)} \mathbf{r}_u^{(t)}}{\tau_u^{(t)} - \nu_u^{(t+1)}}
\end{aligned}$$

and update the scalars by the SE (state-evolution) recursion:

$$\boxed{\begin{aligned}
\chi_u^{(t)} &= \frac{1}{N_u} \mathrm{tr}(\mathbf{A}_u^\dagger \boldsymbol{\Sigma}^{(t)} \mathbf{A}_u) \\
\tau_u^{(t)} &= 1/\chi_u^{(t)} - \lambda_u^{(t)} \\
\nu_u^{(t+1)} &= \frac{1}{N_u} \tau_u^{(t)} \mathbb{E}[\mathrm{tr}(\eta_{u,t}'(\mathbf{s}_u + \sqrt{\tau_u^{(t)}}\, \mathbf{z}))] \\
\lambda_u^{(t+1)} &= \tau_u^{(t)} \nu_u^{(t+1)} / (\tau_u^{(t)} - \nu_u^{(t+1)})
\end{aligned}}$$

Concentration with $\mathcal{L}_p$ norm

- Recall that $\mathbf{y} \in \mathbb{C}^L$ and $\mathbf{s} \in \mathbb{C}^N$ and $\mathbf{s}_u \in \mathbb{C}^{N_u}$. Let U and T be fixed. The scaling parameters are $L, N_1, N_2, \cdots, N_U$ s.t. $L = O(N)$ while N/N_u fixed. Note that this includes scenarios where $L/N \to 0$ as $L, N \to \infty$.
- For $\boldsymbol{\delta} \in \mathbb{C}^d$ where d is arbitrary (e.g. $d = 1$, $d = N$, etc.) we write

$$\boldsymbol{\delta} = \mathcal{O}(1) \quad \text{if} \quad (\mathbb{E}[\|\boldsymbol{\delta}\|^p])^{\frac{1}{p}} \leq C_p$$

  for all $p \in \mathbb{N}$. Also, for any deterministic $\kappa > 0$ (e.g. $\kappa = 1$, $\kappa = 1/\sqrt{N}$)

$$\boldsymbol{\delta} = \mathcal{O}(\kappa) \quad \text{if} \quad \frac{1}{\kappa}\boldsymbol{\delta} = \mathcal{O}(1)$$

- For example, for any small $\epsilon > 0$

$$A = \mathcal{O}(N^{-\epsilon}) \stackrel{a.s.}{\to} 0 \quad \text{as } N \to \infty$$

Main result

Let

- Recall the SVD $\mathbf{A}_u = \mathbf{U}_u \mathbf{\Lambda}_u \mathbf{O}_u$ let each $\mathbf{O}_u$ be an independent Haar unitary.
- $\frac{\|\mathbf{s}_u\|}{\sqrt{N_u}} = \mathcal{O}(1)$, and $\|\mathbf{A}_u\|_2 = \mathcal{O}(1)$.
- The denoisers $\eta_{u,t} : \mathbb{C}^{N_u} \to \mathbb{C}^{N_u}$ uniformly Lipschitz.

Then, under additional technical assumptions, we have

$$\boxed{\mathbf{r}_u^{(t)} = \mathbf{s}_u + \phi_u^{(t)} + \mathcal{O}(1)}$$

where each $\{\phi_u^{(t)} \in \mathbb{C}^{N_u}\}_{t\in[T]}$ is the independent zero-mean Gaussian process that is independent of $\mathbf{s}$ such that $\phi_u^{(t)} \sim_{\text{i.i.d.}} \Phi_u^{(t)}$. In particular, $\mathbb{E}[|\Phi_u^{(t)}|^2] = \tau_u^{(t)}$.

Corollary (Mean-Square Error)

$$\frac{1}{N_u} \|\mathbf{s}_u - \boldsymbol{\eta}_u^{(t+1)}\|^2 = \frac{1}{N_u} \mathbb{E}[\|\mathbf{s}_u - \eta_{u,t}(\mathbf{s}_u + \sqrt{\tau_u^{(t)}}\,\mathbf{z})\|^2] + \mathcal{O}(N^{-\frac{1}{2}})$$

Proof strategy: Random-matrix free equivalent

Lemma (Lu, 2021)

Let $\mathbf{v} \doteq \mathbf{g}/\|\mathbf{g}\|$ *with* $\mathbf{g} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ *and a random vector* $\widetilde{\mathbf{v}} \in \mathbb{C}^N$ *with* $\|\widetilde{\mathbf{v}}\| = 1$ *and a Haar matrix* $\mathbf{O}^{(N-1)} \in \mathbb{C}^{(N-1)\times(N-1)}$ *be mutually independent. Then,*

$$\mathbf{O}^{(N)} \doteq \mathbf{v}\widetilde{\mathbf{v}}^\dagger + \mathbf{\Pi}_{\mathbf{v}}^\perp \mathbf{O}^{(N-1)} (\mathbf{\Pi}_{\widetilde{\mathbf{v}}}^\perp)^\dagger$$

is Haar distributed and independent of $\widetilde{\mathbf{v}}$. *Here, e.g.* $\mathbf{\Pi}_{\widetilde{\mathbf{v}}}^\perp \mathbb{C}^{N\times(N-1)}$ *is a semi-unitary matrix whose columns span the orthogonal complements of the column spans of* $\widetilde{\mathbf{v}}$, *such that* $\mathbf{\Pi}_{\widetilde{\mathbf{v}}}^\perp (\mathbf{\Pi}_{\widetilde{\mathbf{v}}}^\perp)^\dagger = \mathbf{I}_N - \widetilde{\mathbf{v}}\widetilde{\mathbf{v}}^\dagger$. *E.g. let* $\widetilde{\mathbf{v}} \equiv \mathbf{x}/\|\mathbf{x}\|$ *we have*

$$\mathbf{O}^{(N)}\mathbf{x} \equiv \frac{\|\mathbf{x}\|}{\|\mathbf{g}\|}\mathbf{g}$$

Gram-Schmidt notation: Let $\mathbf{v}^{(1:t-1)} = \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(t-1)}\}$ be a set of orthogonal vectors in $\mathbb{C}^N$ as $\langle \mathbf{v}^{(s)}, \mathbf{v}^{(s')} \rangle = \delta_{ss'}$. For any $\mathbf{b} \in \mathbb{C}^N$, we can always construct a new orthogonal vector $\mathbf{v}^{(t)} \doteq \mathcal{GS}(\mathbf{b} \mid \mathbf{v}^{(1:t-1)})$ s.t. $\langle \mathbf{v}^{(t)}, \mathbf{v}^{(s)} \rangle = \delta_{ts}$ and

$$\mathbf{b} = \sum_{1\leq s\leq t} \langle \mathbf{v}^{(s)}, \mathbf{b} \rangle \mathbf{v}^{(s)}$$

Proof strategy: Random-matrix free equivalent

The probability distribution of the sequence $\{\mathbf{r}_u^{(t)}\}_{t\leq T}$ from the original algorithm equals that of the same sequence generated by the following $\mathbf{O}_u$-free dynamics:

$$\begin{aligned}
\widetilde{\phi}_u^{(t)} &= f_{u,t}(\mathbf{r}_u^{(t-1)}) - \mathbf{s}_u \\
\mathbf{v}_u^{(2t-1)} &= \mathcal{GS}(\mathbf{g}_u^{(t)}|\mathbf{v}_u^{(1:2(t-1))}) \\
\widetilde{\mathbf{v}}_u^{(2t-1)} &= \mathcal{GS}(\widetilde{\phi}_u^{(t)}|\widetilde{\mathbf{v}}_u^{(1:2(t-1))}) \\
\mathbf{q}_u^{(t)} &= \sum_{1\leq s<2t} \langle \widetilde{\mathbf{v}}_u^{(s)}, \widetilde{\phi}_u^{(t)} \rangle \mathbf{v}_u^{(s)} \\
\psi_u^{(t)} &= \mathbf{M}_u^{(t)}\mathbf{n} + \mathbf{q}_u^{(t)} - \mathbf{M}_u^{(t)} \sum_{u'} \mathbf{U}_{u'}\mathbf{\Lambda}_{u'}\mathbf{q}_{u'}^{(t)} \\
\mathbf{v}_u^{(2t)} &= \mathcal{GS}(\psi_u^{(t)}|\mathbf{v}_u^{(1:2t-1)}) \\
\widetilde{\mathbf{v}}_u^{(2t)} &= \mathcal{GS}(\widetilde{\mathbf{g}}_u^{(t)}|\widetilde{\mathbf{v}}_u^{(1:2t-1)}) \\
\mathbf{r}_u^{(t)} &= \mathbf{s}_u + \sum_{1\leq s\leq 2t} \langle \mathbf{v}_u^{(s)}, \psi_u^{(t)} \rangle \widetilde{\mathbf{v}}_u^{(s)}
\end{aligned}$$

Here, $\mathbf{g}_u^{(t)}, \widetilde{\mathbf{g}}_u^{(t)} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ is new for each t and for short $\mathbf{M}_u^{(t)} \doteq \frac{1}{\chi_u^{(t)}}\mathbf{\Lambda}_u^{\dagger}\mathbf{U}_u^{\dagger}\mathbf{\Sigma}^{(t)}$.

Proof strategy: Concentration via the notion $\mathcal{O}(\kappa)$

Lemma

Consider the random variables $A = \mathcal{O}(\kappa)$ *and* $B = \mathcal{O}(\widetilde{\kappa})$*. Then,*

$$A + B = \mathcal{O}(\max(\kappa, \widetilde{\kappa}))$$
$$AB = \mathcal{O}(\kappa\widetilde{\kappa})$$

Lemma

Let $f : \mathbb{C}^{3N} \to \mathbb{C}$ *be a pseudo-Lipschitz function, i.e., for every* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{3N}$ *there exists a constant* $C > 0$ *such that*

$$\left|f(\mathbf{a}) - f(\mathbf{b})\right| \leq C\left(1 + \frac{\|\mathbf{a}\|}{\sqrt{N}} + \frac{\|\mathbf{b}\|}{\sqrt{N}}\right)\frac{\|\mathbf{a} - \mathbf{b}\|}{\sqrt{N}}.$$

Let the random vectors $\mathbf{s} \in \mathbb{C}^{N}$ *and* $\mathbf{z} \in \mathbb{C}^{2N}$ *be independent, with* $\|\mathbf{s}\| = \mathcal{O}(\sqrt{N})$ *and* $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{2N})$*. Then,*

$$f(\mathbf{s}, \mathbf{z}) = \mathbb{E}[f(\mathbf{s}, \mathbf{z}) \mid \mathbf{s}] + \mathcal{O}(N^{-\frac{1}{2}}).$$

Part-III: Bayesian Optimality

Results of replica-symmetry ansatz

$$\mathcal{I} \doteq \lim_{N,L\to\infty} \frac{1}{L}\mathbb{E}\Big[\ln \frac{p(\mathbf{y} \mid \mathbf{s}, \mathbf{A})}{p(\mathbf{y} \mid \mathbf{A})} \mid \mathbf{A}\Big]$$

$$\nu_u \doteq \lim_{N,L\to\infty} \frac{1}{N_u}\mathbb{E}\Big[\|\mathbf{s}_u - \mathbb{E}[\mathbf{s}_u \mid \mathbf{y}, \mathbf{A}]\|^2 \mid \mathbf{A}\Big]$$

We have the replica symmetry (RS) predictions of these quantities

$$\mathcal{I} \overset{\mathrm{RS}}{=} \inf_{\boldsymbol{\lambda}\geq \mathbf{0}, \boldsymbol{\tau}>\mathbf{0}} \lim_{N_u,L\to\infty} \frac{1}{L} \ln \Big|\mathbf{I}_L + \frac{1}{\sigma^2}\sum_{u\leq U} \lambda_u \mathbf{A}_u \mathbf{A}_u^\dagger\Big|$$

$$+ \sum_{u\leq U} \alpha_u \Big[\lim_{N_u\to\infty} \frac{1}{N_u}\mathcal{I}\big(\mathbf{s}_u; \mathbf{s}_u + \sqrt{\tau_u}\mathbf{z}\big) - \ln(1 + \frac{\lambda_u}{\tau_u})\Big]$$

$$\nu_u \overset{\mathrm{RS}}{=} \lim_{N_u\to\infty} \frac{1}{N_u}\mathrm{mmse}(\mathbf{s}_u \mid \mathbf{s}_u + \sqrt{\tau_u^\star}\mathbf{z})$$

with $\alpha_u \doteq N_u/L$ fixed. [E.g., with the random semi-unitary $\mathbf{\Xi}_u$, $\mathbf{A}_u\mathbf{A}_u^\dagger = \mathbf{H}_u\mathbf{H}_u^\dagger$]

RS fixed-point equations of $\tau_u^\star$

Let

$$G_u(\boldsymbol{\omega}) \doteq \lim_{N_u, L \to \infty} \frac{1}{N_u} \mathrm{tr}\Big(\mathbf{A}_u^\dagger (\sigma^2 \mathbf{I}_L + \sum_{u' \leq U} \omega_{u'} \mathbf{A}_{u'} \mathbf{A}_{u'}^\dagger)^{-1} \mathbf{A}_u\Big)$$

$$M_u(\rho) \doteq \lim_{N_u \to \infty} \frac{1}{N_u} \mathrm{mmse}(\mathbf{s}_u \mid \mathbf{s}_u + \sqrt{\rho}\mathbf{z})$$

Then,

$$\tau_u^\star = \frac{1}{G_u(\boldsymbol{\lambda}^\star)} - \lambda_u^\star$$

$$\lambda_u^\star = \left(\frac{1}{M_u(\tau_u^\star)} - \frac{1}{\tau_u^\star} \right)^{-1} .$$

Bayesian optimality of the algorithm

Under a regularity condition on the mapping

$$M_{N_u}(\rho) \doteq \frac{1}{N_u}\mathrm{mmse}(\mathbf{s}_u|\mathbf{s}_u + \sqrt{\rho}\mathbf{z})$$

we have the convergence

$$\lim_{t\to\infty}\lim_{N,L\to\infty}\frac{1}{N_u}\|\mathbf{s}_u - \mathbf{s}_u^{(t+1)}\|^2 = \lim_{N_u\to\infty}\frac{1}{N_u}\mathrm{mmse}\left(\mathbf{s}_u \,\middle|\, \mathbf{s}_u + \sqrt{\tau_u^\star}\,\mathbf{z}\right)$$

where $\tau_u^\star$ is a solution of the RS fixed–point equation.

Replica ansatz: Disorder average trick

Let $Z(\mathbf{y}) = \int p(\mathbf{s}, \mathbf{y}) \mathrm{d}\mathbf{s}$. Then, we write

$$\lim_{N_u, L \to \infty} \frac{1}{L} \mathbb{E}_{\{\mathbf{O}_u\}} \int \mathrm{d}\mathbf{y}\; Z(\mathbf{y}) \ln Z(\mathbf{y})$$

$$= \lim_{N, L \to \infty} \lim_{n \to 1} \frac{\partial}{\partial n} \ln \mathbb{E}_{\{\mathbf{O}_u\}} \int \mathrm{d}\mathbf{y}\; Z^n(\mathbf{y})$$

$$\overset{RA}{=} \lim_{n \to 1} \frac{\partial}{\partial n} \lim_{N, L \to \infty} \ln \mathbb{E}_{\{\mathbf{O}_u\}} \int \mathrm{d}\mathbf{y}\; Z^n(\mathbf{y})$$

$$\overset{RA}{=} \lim_{n \to 1} \frac{\partial}{\partial n} \lim_{N, L \to \infty} \ln \int \prod_{a \leq n} \mathrm{dP}(\mathbf{s}^{(a)})\; \mathbb{E}_{\{\mathbf{O}_u\}} \mathrm{e}^{-\mathrm{tr}\left[\left(\sum_u \mathbf{U}_u \boldsymbol{\Lambda}_u \mathbf{O}_u \mathbf{S}_u\right) \mathbf{P}_{\mathbf{1}}^{\perp} \left(\sum_u \mathbf{U}_u \boldsymbol{\Lambda}_u \mathbf{O}_u \mathbf{S}_u\right)^{\dagger}\right]}$$

where $\mathbf{S}_u \doteq [\mathbf{s}_u^{(1)}, \mathbf{s}_u^{(2)}, \ldots, \mathbf{s}_u^{(n)}]$ and $\mathbf{P}_{\mathbf{1}}^{\perp} \doteq \mathbf{I}_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^{\top}$.

For $U = 1$ the disorder average can be computed via the asymptotic Itzykson–Zuber integral.

Replica ansatz: Disorder average trick

- Let $\mathcal{Q}_u \doteq \frac{1}{N_u}\mathbf{S}_u^\dagger \mathbf{S}_u$. Then, we have

$$\mathbf{\Gamma}_u \doteq \mathbf{O}_u\mathbf{S}_u = \sqrt{N_u}\mathbf{V}_u \mathrm{chol}(\mathcal{Q}_u)$$

  where $\mathbf{V}_u \in \mathbb{C}^{N_u \times n}$ random-semi-unitary matrix.

- Hence, conditioned on $\mathcal{Q}_u$,

$$p(\mathbf{\Gamma}_u) = \frac{1}{Z(\mathcal{Q}_u)}\delta\left(\mathbf{\Gamma}_u^\dagger \mathbf{\Gamma}_u - N_u\mathcal{Q}_u\right)$$

  where the normalization constant reads as

$$\ln Z(\mathcal{Q}) = (N_u - n - 1)\ln|N_u\mathcal{Q}| + N_u n \ln \pi - \sum_{i \leq n} \ln \Gamma(N_u - i + 1)$$

- Using this, we represent the disorder average

$$\mathbb{E}_{\{\mathbf{O}_u\}} \mathrm{e}^{-\mathrm{tr}\left[(\sum_u \mathbf{U}_u\mathbf{\Lambda}_u\mathbf{O}_u\mathbf{S}_u)\mathbf{P}_1^\perp(\sum_u \mathbf{U}_u\mathbf{\Lambda}_u\mathbf{O}_u\mathbf{S}_u)^\dagger\right]} = \mathrm{e}^{-\sum_u \ln Z(\mathcal{Q}_u)} \times$$
$$\times \int \prod_u \delta(\mathbf{\Gamma}_u^\dagger\mathbf{\Gamma}_u - N_u\mathcal{Q}_u)\mathrm{d}\mathbf{\Gamma}_\mathrm{u}\, \mathrm{e}^{-\mathrm{tr}\left[(\sum_u \mathbf{U}_u\mathbf{\Lambda}_u\mathbf{\Gamma}_u)\mathbf{P}_1^\perp(\sum_u \mathbf{U}_u\mathbf{\Lambda}_u\mathbf{\Gamma}_u)^\dagger\right]}$$

Simulation results

1. Two users with the asymmetry: $\mathbf{H}_1\boldsymbol{\Xi}_1\mathbf{s}_1 \sim \sqrt{\rho}\mathbf{H}_2\boldsymbol{\Xi}_2\mathbf{s}_2$
2. $\mathbf{H}_1 \sim \sqrt{\rho}\mathbf{H}_2$ are constructed independently with the path parameters $\{\alpha_p, \nu_p, \tau_p\}_{p\leq 23}$ follow from 5G-3GPP (see MATLAB's *5G Toolbox*).
3. We consider an OFDM system with $M = 256$ subcarriers and $n = 16$ OFDM blocks, each of which has a CP, so that

$$\boldsymbol{\Sigma}^{(t)} = (\sigma^2\mathbf{I}_{nM} + \frac{N}{2nM}\sum_u \lambda_u^{(t)}\mathbf{H}_u\mathbf{H}_u^\dagger)^{-1} \quad \text{has } O(nM^3) \text{ complexity.}$$

4. Random semi unitary precodings s.t. $\boldsymbol{\Xi}_u\boldsymbol{\Xi}_u^\dagger = \frac{N}{2nM}\mathbf{I}_{nM}$
5. SPARC–LDPC: The constellations $\mathbf{s}_1 \sim \mathbf{s}_2 \in \mathbb{C}^{Q\times B=256\times 600}$ are created

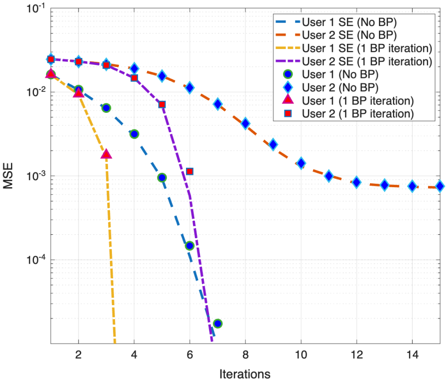
$$\mathbf{s}^\top = \left[\mathbf{s}[1]^\top \mid \mathbf{s}[2]^\top \mid \cdots \mid \mathbf{s}[B]^\top\right] \qquad \mathbf{s}[b] \equiv \mathbf{P}_{\mathrm{LDPC}}\mathbf{e}_1$$

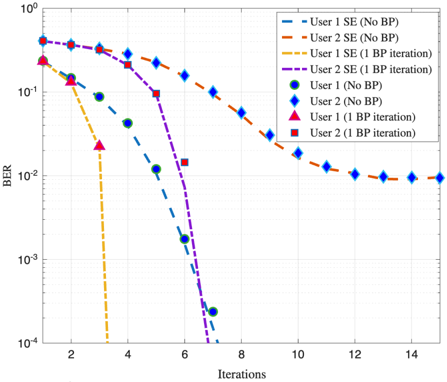
where the permutation is imposed by the non-binary LDPC.

Simulation results

For $b = 1, 2, \cdots, B$ and $q = 1, 2, \cdots, Q$

$$(\eta_0(\mathbf{r};\tau)[b])_q = \mathbb{P}(\mathbf{s}[b] = \mathbf{e}_q \big| \mathbf{r}[b] = \mathbf{s}[b] + \sqrt{\tau}\mathbf{z})$$
$$\eta(\mathbf{r};\tau) = \eta_{\mathrm{BP}}(\eta_0(\mathbf{r};\tau))$$





Time-varying Channels: SR-LDPC signals, $E_b/\sigma^2 = 10.5$ dB, $\rho = 6$dB

Conclusion & Outlook

- The joint multi-user model breaks right-unitary invariance $\Rightarrow$ conventional OAMP/VAMP fails under user statistical asymmetry.
- We introduce a novel multi-user OAMP framework.
- We present a finite-sample analysis for non-separable systems.
- By using the replica-symmetry ansatz, we show that the algorithm achieves Bayesian-optimal performance.
- Verify the universality of high-dimensional analysis for pseudo-random matrices (e.g., random-signed Hadamard) instead of Haar random matrices.
- Investigate the impact of imperfect CSI at the receiver.